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Existence and Monotone Scheme for Time-Periodic Nonquasimonotone Reaction–Diffusion Systems: Application to Autocatalytic Chemistry

Anthony W. Leung*

and

Luis A. Ortega

Department of Mathematics, Universidad Nacional, Bogotá, Colombia

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The article considers time-periodic reaction–diffusion systems. The reaction terms are sums of quasimonotone nondecreasing and nonincreasing functions. Dirichlet and Robin boundary conditions are included. The existence of periodic solutions is shown under appropriate conditions. Monotone approximating sequences closing in on the solutions from above and below are constructed. Application to autocatalysis in chemistry is given. © 1998 Academic Press

1. INTRODUCTION AND EXISTENCE RESULTS

We consider the T-periodic nonlinear boundary value problem,

$$
\begin{aligned}
L[u](x,t) &= F_1(x,t,u,v) + G_1(x,t,u,v) && \text{on } \Omega \times \mathbb{R},\\
L[v](x,t) &= F_2(x,t,u,v) + G_2(x,t,u,v) && \text{on } \Omega \times \mathbb{R},\\
u(x,t) &= u(x,t+T), v(x,t) = v(x,t+T) && \text{on } \Omega \times \mathbb{R},\\
Bu(x,t) &= \psi_1(x,t), Bv(x,t) = \psi_2(x,t) && \text{on } \partial\Omega \times \mathbb{R},
\end{aligned}
\tag{I}
$$

*To whom correspondence should be addressed.

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where Ω is a bounded domain in $\mathbb{R}^N$, $(N \geq 1)$ with boundary $\partial\Omega$ of class $C^{2+\alpha}$ $(0 < \alpha < 1)$, and L is an uniformly parabolic operator defined by

$$L[u](x,t) = \frac{\partial u}{\partial t}(x,t) - A(x,t,D)u(x,t),$$

$$A(x,t,D)u = \sum_{i,j=1}^{N} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i\, \partial x_j}(x,t) + \sum_{i=1}^{N} b_i(x,t)\frac{\partial u}{\partial x_i}(x,t) + c(x,t)u(x,t).$$

The coefficients of L are assumed to be α-Hölder continuous on $\overline{\Omega} \times [0,T]$, T-periodic at the variable t, $a_{ij} \equiv a_{ji}$ and $c \leq 0$ on $\overline{\Omega} \times \mathbb{R}$, $Bu = \alpha_1 u + \beta\, \partial u/\partial\vec{\eta}$, where either (i) $\beta = 0$ and $\alpha_1 = 1$ or (ii) $\beta = 1$ and $\alpha_1 \in C^{\alpha+2}(\overline{\Omega})$, $\alpha_1(x) > 0$, $x \in \partial\Omega$; here $\vec{\eta} = (\eta_1, \dots, \eta_N)$ is the unit outward normal vector field at $\partial\Omega$. The functions ψ_i, F_i, G_i, $i = 1,2$ are T-periodic at the variable t, and $F_i(x,t,s_1,s_2)$, $G_i(x,t,s_1,s_2)$, $i = 1,2$ are continuous on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, with continuous partial derivatives with respect to s_i, $i = 1,2$, on their domain. Furthermore, F_i and G_i are quasimonotone nondecreasing and nonincreasing, respectively, as explained in [H3] below. This system was studied by L. Y. Tsai in [11] with Dirichlet condition on the boundary.

We first establish an existence and uniqueness theorem for the linear case:

$$\begin{aligned} L[u](x,t) &= f(x,t) && \text{on } \Omega \times \mathbb{R}, \\ Bu(x,t) &= \psi(x,t) && \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,t+T) &= u(x,t) && \text{on } \Omega \times \mathbb{R}. \end{aligned} \tag{II}$$

Equation (II) was studied by P. C. Fife in [2] with Dirichlet condition on the boundary. Here we extend the result to more general boundary conditions and then apply the result to study system (I).

In Section 3 we use the results in Section 1 to investigate the following system:

$$\begin{aligned} L[u](x,t) &= av(x,t) - (v(x,t))^2 u(x,t) && \text{on } \Omega \times \mathbb{R}, \\ L[v](x,t) &= v(x,t)^2 u(x,t) + f(x,t) - (a+1)v(x,t) && \text{on } \Omega \times \mathbb{R}, \\ Bu(x,t) &= \psi_1(x,t), Bv(x,t) = \psi_2(x,t) && \text{in } \partial\Omega \times \mathbb{R}, \\ u(x,t) &= u(x,t+T), v(x,t+T) = v(x,t) && \text{in } \overline{\Omega} \times \mathbb{R}, \end{aligned} \tag{V}$$

where a is a positive constant, $f \in \hat{A}, c(x,t) < 0$, and $\psi_i \in \hat{E}$, $\psi_i \geq 0$, $f > 0$, $\psi_i \not\equiv 0$, $i = 1,2$ in $\overline{\Omega} \times \mathbb{R}$. The operator B is defined above, and $\hat{A}$ and $\hat{E}$ are t-periodic functions defined in the next paragraph. System (V) is related to autocatalytic chemical reactions. It is also known as the Brusselator problem, a model for chemical morphogenetic process due to Turing [12]. This system was studied for the case of the initial boundary value problem in [5] and [8]. In Section 4, we construct a scheme for approximating the periodic solution for problem (I). Monotonic sequences of periodic functions are constructed converging to upper and lower bounds of the solution. Since every smooth function can be written as the sum of a nondecreasing function and a nonincreasing function, the scheme is useful for analyzing a more extensive class of problems than it seems in the right side of (I). The scheme can also readily be used for the elliptic case, and the method can be used for finding conditions for uniqueness in some cases (see, e.g., Section 5.3 in [7]).

Throughout this paper all functions are real-valued. We denote by

$$\hat{A} = \left\{u \in C^{\alpha, \alpha/2}(\overline{\Omega} \times \mathbb{R}) | u(x,t) = u(x,t+T) \text{ on } \overline{\Omega} \times \mathbb{R}\right\},$$
$$\text{with norm } \|u\|_{\hat{A}} = \|u\|_{C^{\alpha,\alpha/2}(\overline{\Omega} \times \mathbb{R})},$$
$$\hat{E} = \left\{u \in C^{2+\alpha, 1+\alpha/2}(\overline{\Omega} \times \mathbb{R}) | u(x,t) = u(x,t+T) \text{ on } \overline{\Omega} \times \mathbb{R}\right\},$$
$$\text{with norm } \|u\|_{\hat{E}} = \|u\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times \mathbb{R})}.$$

We assume:

[H1] The functions $F_i(\cdot, u, v)$, $G_i(\cdot, u, v)$, $\partial F_i / \partial s_j(\cdot, u, v)$, $\partial G_i / \partial s_j(\cdot, u, v) \in \hat{A}$, $i, j = 1,2$, for $u, v \in \hat{A}$ and the functions $\psi_i \in \hat{E}$, for $i = 1,2$

[H2] There are functions $\overline{u}, \overline{v}, \underline{u}, \underline{v} \in \hat{E}$, such that

$$\begin{aligned}
L[\overline{u}](x,t) &\geq F_1(x,t,\overline{u},\overline{v}) + G_1(x,t,\overline{u},\underline{v}) \qquad \text{on } \Omega \times \mathbb{R},\\
L[\underline{u}](x,t) &\leq F_1(x,t,\underline{u},\underline{v}) + G_1(x,t,\underline{u},\overline{v}) \qquad \text{on } \Omega \times \mathbb{R},\\
L[\overline{v}](x,t) &\geq F_2(x,t,\overline{u},\overline{v}) + G_2(x,t,\underline{u},\overline{v}) \qquad \text{on } \Omega \times \mathbb{R},\\
L[\underline{v}](x,t) &\leq F_2(x,t,\underline{u},\underline{v}) + G_2(x,t,\overline{u},\underline{v})
\end{aligned}$$
$$B\overline{u} \geq \psi_1, \quad B\underline{u} \leq \psi_1, \quad B\overline{v} \geq \psi_2, \quad B\underline{v} \leq \psi_2 \qquad \text{on } \partial\Omega \times \mathbb{R},$$
$$\overline{u} \geq \underline{u}, \quad \overline{v} \geq \underline{v} \qquad \text{on } \Omega \times \mathbb{R}.$$

[H3] The functions F_1 and F_2 are quasimonotone nondecreasing and G_1 and G_2 are quasimonotone nonincreasing in the set

$$S = \{(x,t,u,v) | (x,t) \in \overline{\Omega} \times \mathbb{R}, \underline{u}(x,t) \leq u \leq \bar{u}(x,t), \quad \underline{v}(x,t) \leq v \leq \bar{v}(x,t)\}.$$

Here the function $F_i(x,t,u_1,u_2)$ is said to be *quasimonotone nondecreasing* (resp. *nonincreasing*) in some subset $\tilde{S}$ of $\Omega \times \mathbb{R} \times \mathbb{R}^2$, if for fixed u_i with (x,t,u_1,u_2) in $\tilde{S}$ for all $x \in \overline{\Omega}$, $t \in \mathbb{R}$, F_i is nondecreasing (resp. nonincreasing) in u_j, $j \neq i$, $i,j = 1,2$.

THEOREM 1.1. *For each $f \in \hat{A}$ and $\psi \in \hat{E}$, there exists a unique solution $u \in \hat{E}$ of* (II). *Furthermore, there exists a constant $\hat{k} > 0$, independent of f and ψ, and a constant $c_2 = c_2(\psi) \geq 0$, $c_2(0) = 0$, such that*

$$\|u\|_{\hat{E}} \leq \hat{k}\left[\|f\|_{\hat{A}} + c_2(\psi) + \|\psi\|_\infty^{\overline{\Omega}|x[0,T]}\right]. \tag{1.1}$$

More precisely,

$$c_2(\psi) \leq \bar{k}\|\psi\|_{C^{2+\alpha,1+\alpha/2}(\partial\Omega\times[0,T])}$$

for boundary condition of case (i) *and*

$$c_2(\psi) \leq \bar{k}\|\psi\|_{C^{1+\alpha,(1+\alpha)/2}(\partial\Omega\times[0,T])}$$

for boundary condition of case (ii), *where $\bar{k} > 0$ is a constant.*

Remark 1.1. For boundary condition of case (ii), the smoothness assumption of ψ may be relaxed to be inside the class

$$C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega} \times [0,T]).$$

THEOREM 1.2. *If* [H1–H3] *hold, then there exists a solution $(u,v), u,v \in \hat{E}$ of* (I) *such that*

$$\underline{u} \leq u \leq \bar{u} \quad \text{and} \quad \underline{v} \leq v \leq \bar{v}, \qquad \text{on } \overline{\Omega} \times \mathbb{R}. \tag{1.2}$$

2. PROOF OF EXISTENCE RESULTS

In the proof of Theorem 1.1, we use the following lemma:

LEMMA 2.1. *For each $\psi \in \hat{E}$, $\psi \geq 0$ on $\partial\Omega \times \mathbb{R}$, there exists a solution $\chi(\psi) \in \hat{E}$ of the T-periodic linear problem*:

$$\begin{aligned} L[u] &= 0 \qquad \text{on } \Omega \times \mathbb{R}, \\ Bu(x,t) &= \psi(x,t) \qquad \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,t) &= u(x,t+T) \qquad \text{on } \overline{\Omega} \times \mathbb{R}. \end{aligned} \tag{III}$$

Moreover, there exist constants $c_1 > 0$ *independent of* ψ *and* $c_2(\psi) \geq 0$, $c_2(0) = 0$ *such that* $\chi(\psi)$ *satisfies the estimates*

$$\|\chi(\psi)\|_{C^{2+\alpha,1+\alpha/2}(\Omega\times[0,T])} \leq c_1\left[\|\psi\|_\infty^{\overline{\Omega}\times[0,T]} + c_2(\psi)\right]. \tag{2.1}$$

More precise bounds on $c_2(\psi)$ *are as described at the end of Theorem* 1.1 *and Remark* 1.1.

Proof. For each $0 < \delta < T$, let $h_\delta \in C^\infty(\mathbb{R},\mathbb{R})$ be a function such that, $h_\delta(t) = 0$ if $t \leq \delta/2$ and $h_\delta(t) = 1$ if $t \geq \delta$ and $0 \leq h_\delta(t) \leq 1$ for all $t \in \mathbb{R}$. By a standard result ([6], Chap. 4, Theorems 5.2 and 5.3), there exists a unique solution $w \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,r])$, for all $r > 0$, of the linear initial boundary value problem:

$$\begin{aligned} L[u] &= 0 \qquad \text{on } \Omega \times [0,\infty), \\ Bu(x,t) &= \psi(x,t)h_\delta(t) \qquad \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) &= 0 \qquad \text{on } \overline{\Omega}. \end{aligned} \tag{IV}$$

Moreover, $w \geq 0$ on $\Omega \times [0,\infty)$.

We will prove that w is a bounded function on $\overline{\Omega} \times [0,\infty)$. Let

$$k^* = \|\psi\|_\infty^{\overline{\Omega}\times[0,T]}\left(\inf\{\alpha_1(x),\ x \in \partial\Omega\}\right)^{-1}. \tag{2.2}$$

Since the function $(k^* - w)$ satisfies

$$\begin{aligned} L[v] &= -c(x,t)k^* \geq 0, \qquad \text{on } \Omega \times [0,\infty), \\ Bv(x,t) &= \alpha_1(x)k^* - h_\delta(t)\psi(x,t) \geq 0 \qquad \text{on } \partial\Omega \times [0,\infty), \\ v(x,0) &= k^* \qquad \text{on } \overline{\Omega}, \end{aligned} \tag{2.3}$$

it follows from the Maximum Principle that

$$0 \leq w(x,t) \leq k^* = \|\psi\|_\infty^{\overline{\Omega}\times[0,T]}\left(\inf\{\alpha_1(x),\ x \in \partial\Omega\}\right)^{-1}. \tag{2.4}$$

Let $\hat{w}(x,t) = w(x,t+T) - w(x,t)$ on $\overline{\Omega} \times [0,\infty)$. Since $\hat{w}$ satisfies the problem

$$\begin{aligned} L[z] &= 0 \qquad \text{on } \Omega \times [0,\infty), \\ Bz(x,t) &= \psi(x,t) - h_\delta(t)\psi(x,t) \geq 0 \qquad \text{on } \partial\Omega \times [0,\infty), \\ z(x,0) &= w(x,T) \geq 0 \qquad \text{on } \overline{\Omega}, \end{aligned}$$

it follows that $0 \leq w(x,t) \leq w(x,t+T)$ on $\overline{\Omega} \times [0,\infty)$. We set $w_m(x,t) = w(x,t+mT)$ for all positive integers m, $(x,t) \in \overline{\Omega} \times [0,\infty)$.

Notice that

$$L[w_m] = 0 \qquad \text{on } \Omega \times [0, \infty),$$
$$Bw_m(x, t) = \psi(x, t) \qquad \text{on } \partial\Omega \times [0, \infty),$$
$$w_m(x, 0) = w(x, mT) \geq 0 \qquad \text{on } \overline{\Omega}.$$

Since

$$0 \leq w_m(x, t) = w(x, t + mT) \leq w(x, t + T + mT) = w_{m+1}(x, t) \leq k^*,$$

there exists a function $\tilde{w}$ defined on $\overline{\Omega} \times [0, \infty)$ such that

$$0 \leq \lim_{m \to \infty} w_m(x, t) = \tilde{w}(x, t) \leq c\|\psi\|_\infty^{\overline{\Omega} \times [0, T]}$$
$$\text{for all } (x, t) \in \overline{\Omega} \times [0, \infty).$$

and

$$\|\tilde{w}(\cdot, t) - w_n(\cdot, t)\|_{L_p(\Omega)} \to 0, \qquad \text{as } n \to \infty \text{ for all } t \in [0, \infty).$$

Moreover, $\tilde{w}$ is a T-periodic function; in fact,

$$\tilde{w}(x, t + T) = \lim_{m \to \infty} w_{m+1}(x, t) = \tilde{w}(x, t) \qquad \text{on } \overline{\Omega} \times [0, \infty).$$

Notice that if we prove that the sequence $\{w_m\}$ converges to $\tilde{w}$ in the Hölder space $C^{2+\alpha, 1+\alpha/2}(\overline{\Omega} \times [r/2, r])$ for all $r \geq 2T$, it follows from the periodicity of $\tilde{w}$ that this function is a solution of (III).

Let m and n be fixed positive integers. Since $(w_n - w_m)$ satisfies the linear equation

$$L[z] = 0 \qquad \text{on } \Omega \times [0, \infty),$$
$$Bz(x, t) = 0 \qquad \text{on } \partial\Omega \times [0, \infty),$$
$$z(x, 0) = w_n(x, 0) - w_m(x, 0) \qquad \text{on } \overline{\Omega},$$

we have $(w_n - w_m)(\cdot, t) = \mathscr{U}(t, 0)(w_n(\cdot, 0) - w_m(\cdot, 0))$, where $\mathscr{U}(s, t)$ is the Evolution System associated with the operator:

$$A(t)u = \sum_{i,j=1}^{N} a_{ij}(\cdot, t) \frac{\partial^2 u}{\partial x_i \, \partial x_j}(\cdot, t) + \sum_{i=1}^{N} b_i(\cdot, t) \frac{\partial u}{\partial x_i}(\cdot, t) + c(\cdot, t)u,$$

$$D(A(t)) = \{u \in W^{2,p}(\Omega) | Bu = 0 \text{ on } \partial\Omega\}.$$

It follows from a standard result ([1], Lemma 2.1, Proposition 4.1) that

$$\|(w_m - w_n)(\cdot, t)\|_{1+\lambda}^{\overline{\Omega}} \leq t_0^{-\sigma} c(r) \|w_m(\cdot, 0) - w_n(\cdot, 0)\|_{L_p(\Omega)}, \quad (2.5)$$

where $0 < t_0 \leq t$, $\sigma \in (\beta, 1)$, $\beta \in (0, 1)$, $1/2 + N/(2p) < \beta < 1$, $0 < \lambda < 2\beta - 1 - N/p$, and $c(r) > 0$. (We can take $T/5 \leq t_0 \leq T/4$.)

From (2.5), we obtain

$$\|(w_m - w_n)\|_{\infty}^{\overline{\Omega} \times [T/4, r]} \leq t_0^{-\sigma} c(r) \|w_m(\cdot, 0) - w_n(\cdot, 0)\|_{L_p(\Omega)} \to 0, \quad \text{as } m, n \to \infty. \quad (2.6)$$

Let $v(x, t) = h(t)(w_m(x, t) - w_n(x, t))$ on $\overline{\Omega} \times [0, \infty)$, where $h \in C^{\infty}(\mathbb{R}, \mathbb{R})$, $h(t) = 0$, for $t \leq T/4$, $h(t) = 1$ if $t \geq T/2$. The function $v(x, t)$ satisfies the linear equation

$$\begin{aligned} L[u](x, t) &= h'(t)\big(w_m(x, t) - w_n(x, t)\big) \qquad \text{on } \Omega \times [0, \infty), \\ Bu(x, t) &= 0 \qquad \text{on } \partial\Omega \times [0, \infty), \\ u(x, 0) &= 0 \qquad \text{on } \overline{\Omega}. \end{aligned}$$

It follows ([6], Chap. 4, Theorems 5.2 and 5.3) that

$$\begin{aligned} \|h(w_m - w_n)\|_{2+\alpha}^{\overline{\Omega} \times [0, r]} &\leq \hat{c}(r) \|h'(w_m - w_n)\|_{\alpha}^{\overline{\Omega} \times [0, r]} \\ &= \hat{c}(r) \|h'(w_m - w_n)\|_{\alpha}^{\overline{\Omega} \times [T/4, r]}. \end{aligned} \quad (2.7)$$

Since $w^* = (w_m - w_n)h'$ satisfies

$$\begin{aligned} L[u](x, t) &= h''(t)\big(w_m(x, t) - w_n(x, t)\big) \qquad \text{on } \Omega \times [0, \infty), \\ Bu(x, t) &= 0 \qquad \text{on } \partial\Omega \times [0, \infty), \\ u(x, 0) &= 0 \qquad \text{on } \overline{\Omega}, \end{aligned}$$

it follows that

$$w^*(\cdot, t) = \int_0^t \mathscr{U}(t, s) h''(s)\big(w_m(\cdot, s) - w_n(\cdot, s)\big) ds;$$

and by a standard result ([1], Corollary 2.2), we have $w^* \in C^{\gamma}([0, r], X_\beta)$, for $\beta \in (0, 1)$, $1/2 + N/(2p) < \beta < 1$, $\gamma \in (0, 1 - \beta)$, and

$$\|w^*\|_{C^{\gamma}([0, r], X_\beta)} \leq c(\gamma, \beta) \max_{T/4 \leq s \leq r} \|w_m(\cdot, s) - w_n(\cdot, s)\|_{L_p(\Omega)}, \ r > 0.$$

This inequality implies

$$\sup_{0 \le t,\, t' \le r,\, t \ne t'} \frac{\|w^*(\cdot, t) - w^*(\cdot, t')\|_{X_\beta}}{|t - t'|^\gamma} \le c(\gamma, \beta) \max_{T/4 \le s \le r} \|w_m(\cdot, s) - w_n(\cdot, s)\|_{L_p(\Omega)}.$$

Since $X_\beta \hookrightarrow C^{1+\sigma}(\overline{\Omega})$ for $\beta \in (0,1)$, $1/2 + N/(2p) < \beta < 1$, $0 < \sigma < 2\beta - 1 - N/p$ ([1], Proposition 4.1), by the last inequality we obtain

$$\begin{aligned}\sup_{0 \le t,\, t' \le r,\, t \ne t'} & \frac{\|w^*(\cdot, t) - w^*(\cdot, t')\|_{C^{1+\sigma}(\overline{\Omega})}}{|t - t'|^\gamma} \\ &\le \hat{k} \sup_{0 \le t,\, t' \le r,\, t \ne t'} \frac{\|w^*(\cdot, t) - w^*(\cdot, t')\|_{X_\beta}}{|t - t'|^\gamma} \\ &\le \hat{k} c(\gamma, \beta) \max_{T/4 \le s \le r} \|w_m(\cdot, s) - w_n(\cdot, s)\|_{L_p(\Omega)}.\end{aligned} \tag{2.8}$$

Let $\theta = 2\gamma$. Note that $0 < \gamma < 1 - \beta < \frac{1}{2}$ and $0 < \theta < 1$. Since

$$\begin{aligned}\frac{|w^*(x, t) - w^*(x', t')|}{\left[\|x - x'\|^2 + |t - t'|\right]^{\theta/2}} &\le \frac{|w^*(x, t) - w^*(x', t)|}{\|x - x'\|^\theta} \\ &\quad + \frac{|w^*(x', t) - w^*(x', t')|}{|t - t'|^\gamma} \\ &\le \|w^*(\cdot, t)\|_{C^\theta(\overline{\Omega})} + \frac{|w^*(x', t) - w^*(x', t')|}{|t - t'|^\gamma},\end{aligned} \tag{2.9}$$

by (2.5), (2.8), and (2.9), we have

$$\begin{aligned}\sup_{(x,t),\,(x',t') \in \overline{\Omega} \times [0,r],\,(x,t) \ne (x',t')} & \frac{|w^*(x, t) - w^*(x', t')|}{\left[\|x - x'\|^2 + |t - t'|\right]^{\theta/2}} \\ \le k \Big(& \sup_{T/4 \le s \le r} \|w_m(\cdot, s) - w_n(\cdot, s)\|_{L_p(\Omega)} \\ & + \|w_m(\cdot, 0) - w_n(\cdot, 0)\|_{L_p(\Omega)} \Big) \to 0 \qquad \text{as } m, n \to \infty.\end{aligned} \tag{2.10}$$

From (2.6) and (2.10), we obtain

$$
\begin{aligned}
&\|h'(w_m - w_n)\|_{C^{\theta,\theta/2}(\overline{\Omega}\times[0,r])} \\
&\quad \le k\Big(\sup_{T/4\le s\le r} \|w_m(\cdot,s) - w_n(\cdot,s)\|_{L_p(\Omega)} \\
&\qquad\qquad + \|w_m(\cdot,0) - w_n(\cdot,0)\|_{L_p(\Omega)}\Big) \to 0 \qquad \text{as } m,n\to\infty.
\end{aligned}
$$

From a standard result ([6], Chap. 4, Theorems 5.2 and 5.3) and the above inequality,

$$
\begin{aligned}
&\|h(w_m - w_n)\|_{C^{2+\theta,1+\theta/2}(\overline{\Omega}\times[0,r])} \\
&\quad = \|h(w_m - w_n)\|_{C^{2+\theta,1+\theta/2}(\overline{\Omega}\times[T/4,r])} \\
&\quad \le \|h'(w_m - w_n)\|_{C^{\theta,\theta/2}(\overline{\Omega}\times[0,r])} \to 0 \qquad \text{as } m,n\to\infty. \qquad (2.11)
\end{aligned}
$$

With the same arguments as above, we can further prove that

$$
\begin{aligned}
&\|h'(w_m - w_n)\|_{C^{2+\theta,1+\theta/2}(\overline{\Omega}\times[0,r])} \\
&\quad \le c\|h''(w_m - w_n)\|_{C^{\theta,\theta/2}(\overline{\Omega}\times[0,r])} \\
&\quad = c\|h''(w_m - w_n)\|_{C^{\theta,\theta/2}(\overline{\Omega}\times[T/4,r])} \to 0 \qquad \text{as } m,n\to\infty. \qquad (2.12)
\end{aligned}
$$

From (2.11) and (2.12), we have for any $\alpha \in (0,1)$,

$$
\begin{aligned}
&\|h(w_m - w_n)\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,r])} \\
&\quad \le k_1\|h'(w_m - w_n)\|_{C^{\alpha,\alpha/2}\,(\overline{\Omega}\times[0,r]))} \\
&\quad \le c^*\|h'(w_m - w_n)\|_{C^{2+\theta,1+\theta/2}(\overline{\Omega}\times[T/4,r])} \to 0 \qquad \text{as } m,n\to\infty. \qquad (2.13)
\end{aligned}
$$

Thus, from (2.12) and (2.13), we obtain

$$
\begin{aligned}
&\|(w_m - w_n)\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[T/2,r])} \\
&\quad \le \|h(w_m - w_n)\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,r])} \\
&\quad \le \tilde{k}\|w_m(\cdot,0) - w_n(\cdot,0)\|_{L_p(\Omega)} \to 0 \qquad \text{as } m,n\to\infty. \qquad (2.14)
\end{aligned}
$$

For $r > 2T$, it follows from (2.14) that

$$
\|(\tilde{w} - w_n)\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[T/2,2T])} \to 0 \qquad \text{as } m,n\to\infty,
$$

and by the periodicity of $\tilde{w}$, $\tilde{w}$ is a solution of (III).

To prove the estimate (2.1), we consider the inequality (2.14) for $n = 1$ and $r = 2T$, that is,

$$\|(w_m - w_1)\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[T/2,2T])} \le \tilde{k}\|w_m(\cdot,0) - w_1(\cdot,0)\|_{L_p(\Omega)}. \quad (2.15)$$

Letting m tend to ∞ in (2.15), we have

$$\|(\tilde{w} - w_1)\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[T/2,2T])} \le \tilde{k}\|\tilde{w}(\cdot,0) - w_1(\cdot,0)\|_{L_p(\Omega)}. \quad (2.16)$$

From (2.4), (2.16), and the T-periodicity of $\tilde{w}$, we obtain the estimate of (2.1), where $w_1(x,t) = w(x, t + T)$; w is the solution of (IV), and

$$c_2(\psi) = \|w_1\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,2T])}.$$

Proof of Theorem 1.1. By a standard result ([4], Lemma 1.14) for $f \in \hat{A}$, there exists a unique solution $\hat{w}_1 \in \hat{E}$ of the linear T-periodic problem,

$$\begin{aligned} L[u] &= f && \text{on } \Omega \times \mathbb{R}, \\ Bu &= 0 && \text{on } \partial\Omega \times \mathbb{R}, \\ u(x, t + T) &= u(x,t) && \text{on } \overline{\Omega} \times \mathbb{R}, \end{aligned} \quad (2.17)$$

and

$$\|\hat{w}_1\|_{\hat{E}} \le c\|f\|_{\hat{A}}. \quad (2.18)$$

Step 1. We first consider the case $\psi \ge 0$ on $\partial\Omega \times \mathbb{R}$. Then by Lemma 1, the function $u = \hat{w}_1 + \chi(\psi)$ is a solution of (II), where $\hat{w}_1$ is the solution of (2.17) and $\chi(\psi)$ is the solution of (III).

Step 2. We now consider an arbitrary $\psi \in \hat{E}$. Let $\bar{k}$ be large enough such that $\psi + \bar{k}\alpha_1 \ge 0$ on $\partial\Omega \times [0, T]$. By Step 1, there exists a unique solution $\hat{u} \in \hat{E}$ of the T-periodic linear problem,

$$\begin{aligned} L[u] &= f - c(x,t)\bar{k} && \text{on } \Omega \times \mathbb{R}, \\ Bu &= \psi + \bar{k}\alpha_1 && \text{on } \partial\Omega \times \mathbb{R}, \\ u(x, t + T) &= u(x,t) && \text{on } \overline{\Omega} \times \mathbb{R}. \end{aligned} \quad (2.19)$$

The function $\bar{u} = \hat{u} - \bar{k}$ is the unique solution of (II). The estimate (1.1) follows from (2.1) and (2.18).

Proof of Theorem 1.2. Denote

$$M = \sup\left\{\|\bar{u}\|_{\hat{E}}, \|\bar{v}\|_{\hat{E}}, \|\underline{u}\|_{\hat{E}}, \|\underline{v}\|_{\hat{E}}\right\},$$

and

$$K = \sup\left\{\left|\frac{\partial F_i}{\partial s_j}(x,t,s_1,s_2)\right|, \left|\frac{\partial G_i}{\partial s_j}(x,t,s_1,s_2)\right| : (x,t) \in \overline{\Omega} \times \mathbb{R},\right.$$
$$\left. -M \le s_i \le M,\ i,j = 1,2\right\}.$$

Define

$$\sigma_1 u(x,t) = \begin{cases} \bar{u}(x,t) & \text{if } \bar{u}(x,t) \le u(x,t), \\ u(x,t) & \text{if } \underline{u}(x,t) \le u(x,t) \le \bar{u}(x,t), \\ \underline{u}(x,t) & \text{if } u(x,t) \le \underline{u}(x,t), \end{cases}$$

for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$, $u \in \hat{A}$,

$$\sigma_2 v(x,t) = \begin{cases} \bar{v}(x,t) & \text{if } \bar{v}(x,t) \le v(x,t), \\ v(x,t) & \text{if } \underline{v}(x,t) \le v(x,t) \le \bar{v}(x,t), \\ \underline{v}(x,t) & \text{if } v(x,t) \le \underline{v}(x,t), \end{cases}$$

for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$, $v \in \hat{A}$.

We consider the auxiliary problem:

$$\begin{aligned} L[u] + 2Ku &= F_1(\cdot,\cdot,\sigma_1 u, \sigma_2 v) + G_1(\cdot,\cdot,\sigma_1 u, \sigma_2 v) \\ &\quad + 2K\sigma_1 u \qquad \text{on } \Omega \times \mathbb{R}, \\ L[v] + 2Kv &= F_2(\cdot,\cdot,\sigma_1 u, \sigma_2 v) + G_2(\cdot,\cdot,\sigma_1 u, \sigma_2 v) \\ &\quad + 2K\sigma_2 v \qquad \text{on } \Omega \times \mathbb{R}, \end{aligned} \tag{2.20}$$
$$u(x,t) = u(x,t+T), \quad v(x,t) = v(x,t+T) \qquad \text{on } \Omega \times \mathbb{R},$$
$$Bu(x,t) = \psi_1(x,t), \quad Bv(x,t) = \psi_2(x,t) \quad \text{on } \partial\Omega \times \mathbb{R}.$$

For each $u, v \in \hat{A}$ we denote by $\Phi(u,v) \in \hat{E} \times \hat{E}$ the unique solution of the linear system

$$\begin{aligned} L[Z_1] + 2KZ_1 &= F_1(\cdot,\cdot,\sigma_1 u, \sigma_2 v) + G_1(\cdot,\cdot,\sigma_1 u, \sigma_2 v) \\ &\quad + 2K\sigma_1 u \qquad \text{on } \Omega \times \mathbb{R}, \\ L[Z_2] + 2KZ_2 &= F_2(\cdot,\cdot,\sigma_1 u, \sigma_2 v) + G_2(\cdot,\cdot,\sigma_1 u, \sigma_2 v) \\ &\quad + 2K\sigma_2 v \qquad \text{on } \Omega \times \mathbb{R}, \end{aligned} \tag{2.21}$$
$$Z_1(x,t) = Z_1(x,t+T), \quad Z_2(x,t) = Z_2(x,t+T) \qquad \text{on } \Omega \times \mathbb{R},$$
$$BZ_1(x,t) = \psi_1(x,t), \quad BZ_2(x,t) = \psi_2(x,t) \qquad \text{on } \partial\Omega \times \mathbb{R}.$$

Using estimate (1.1) of Theorem 1.1 and the method of the proof of Lemma 2.1, we can prove that the operator Φ from $\hat{A} \times \hat{A}$ to $\hat{A} \times \hat{A}$ is a completely continuous operator and there exists $r > 0$ such that

$$\|\Phi(u,v)\|_{\hat{A}\times\hat{A}} \le r, \qquad \text{for all } (u,v) \in \hat{A} \times \hat{A}. \tag{2.22}$$

For an outline of the proof of (2.22), consider the problem

$$\begin{aligned} L[\hat{Z}] + 2K\hat{Z} = \big[&F_1(x,t,\sigma_1 u, \sigma_2 v) \\ &+G_1(x,t,\sigma_1 u,\sigma_2 v) + 2K\sigma_1 u\big] h_\delta(t) \\ &\text{in } \Omega \times [0,\infty), \\ B\hat{Z}(x,t) = 0 \qquad &\text{in } \partial\Omega \times [0,\infty), \\ \hat{Z}(x,0) = 0 \qquad &\text{in } \overline{\Omega}, \end{aligned} \tag{2.23}$$

where $h_\delta(t)$ is as described in the proof of Lemma 2.1. From the boundedness on the right-hand side of (2.23), we obtain a uniform bound on the $L_p(\Omega)$-norm of $\hat{Z}(x,t)$ for all $t \ge 0$. Then consider the functions $Z_m(x,t) = \hat{Z}(x, t+mT)$ for all positive integers m, $(x,t) \in \overline{\Omega} \times [0,\infty)$. We have

$$\begin{aligned} L[Z_m - Z_n] = 0 \qquad &\text{on } \Omega \times [0,\infty), \\ B(Z_m - Z_n) = 0 \qquad &\text{on } \partial\Omega \times [0,\infty), \\ (Z_m - Z_n)(x,0) = \hat{Z}(x,mT) - \hat{Z}(x,nT) \qquad &\text{on } \overline{\Omega}. \end{aligned} \tag{2.24}$$

From (2.24), we obtain

$$\|(Z_m - Z_n)(\cdot,t)\|_{1+\lambda}^{\overline{\Omega}} \le k_1 \|(Z_m - Z_n)(\cdot,0)\|_{L_p(\Omega)},$$

as in (2.5), where λ is as described there. Furthermore, using the fact that $h_\delta(t)[(Z_m - Z_n)(x,t)]$ satisfies

$$\begin{aligned} L\big[h_\delta[(Z_m - Z_n)]\big](x,t) = h'_\delta(t)(Z_m - Z_n)(x,t) \qquad &\text{on } \Omega \times [0,\infty), \\ B\big[h_\delta(t)[(Z_m - Z_n)]\big] = 0 \qquad &\text{on } \partial\Omega \times [0,\infty), \\ h_\delta(Z_m - Z_n)(x,0) = 0 \qquad &\text{on } \overline{\Omega}, \end{aligned} \tag{2.25}$$

we then show, as in (2.9) and (2.10), that

$$\begin{aligned} &\|Z_m - Z_n\|_{C^{2\gamma,\gamma}(\overline{\Omega}\times[0,2T])} \\ &\quad \le k_2 \Big[\sup_{T/4 \le s \le 2T} \|(Z_m - Z_n)(\cdot,s)\|_{L_p(\Omega)} + \|(Z_m - Z_n)(\cdot,0)\|_{L_p(\Omega)} \Big]. \end{aligned} \tag{2.26}$$

Here γ satisfies $0 < \gamma < \min\{1 - \beta, \beta - N/(2p)\}$, and β is any number satisfying $1/2 + N/(2p) < \beta < 1$. Thus, if we choose $p > 0$ large enough, γ can be any number in $(0, \frac{1}{2})$. In (2.26), fix $n = 1$, and let $m \to \infty$; we deduce a bound for $\|Z\|_{C^{2\gamma,\gamma}(\overline{\Omega}\times[0,T])}$, where $Z(x,t)$ is the solution of

$$\begin{aligned} L[Z] + 2KZ &= F_1(x,t,\sigma_1 u, \sigma_2 v) \\ &\quad + G_1(x,t,\sigma_1 u,\sigma_2 v) + 2K\sigma_1 u, \qquad \text{in } \Omega \times [0,\infty) \\ BZ(x,t) &= 0 \qquad \text{in } \partial\Omega \times [0,\infty), \\ Z(x,t) &= Z(x,t+T) \qquad \text{in } \overline{\Omega} \times \mathbb{R}, \end{aligned} \tag{2.27}$$

as in the proof of Lemma 2.1. Applying Lemma 2.1 again for the given boundary data ψ_1, we obtain a bound for $\|Z_1\|_{\hat{A}}$, $0 < \alpha < 1$. Similar treatment for $Z_2(x,t)$ in (2.21) leads to (2.22).

Let $\mathscr{B}$ denote the open ball in $\hat{A} \times \hat{A}$ centered at $(0,0)$ with radius $r > 0$. Then by (2.22), $\Phi(\overline{\mathscr{B}}) \subseteqq \overline{\mathscr{B}}$. It follows from Schauder's Fixed-Point Theorem that there exists $(\tilde{u},\tilde{v}) \in \overline{\mathscr{B}}$, such that $\Phi((\tilde{u},\tilde{v})) = (\tilde{u},\tilde{v})$; that is, we have

$$\begin{aligned} L[\tilde{u}] + 2K\tilde{u} &= F_1(\cdot,\cdot,\sigma_1\tilde{u},\sigma_2\tilde{v}) + G_1(\cdot,\cdot,\sigma_1\tilde{u},\sigma_2\tilde{v}) \\ &\quad + 2K\sigma_1\tilde{u} \qquad \text{on } \Omega\times\mathbb{R}, \\ L[\tilde{v}] + 2K\tilde{v} &= F_2(\cdot,\cdot,\sigma_1\tilde{u},\sigma_2\tilde{v}) + G_2(\cdot,\cdot,\sigma_1\tilde{u},\sigma_2\tilde{v}) \\ &\quad + 2K\sigma_2\tilde{v} \qquad \text{on } \Omega\times\mathbb{R}, \\ \tilde{u}(x,t) &= \tilde{u}(x,t+T), \quad \tilde{v}(x,t) = \tilde{v}(x,t+T) \qquad \text{on } \Omega\times\mathbb{R}, \\ B\tilde{u}(x,t) &= \psi_1(x,t), \quad B\tilde{v}(x,t) = \psi_2(x,t) \qquad \text{on } \partial\Omega\times\mathbb{R}. \end{aligned} \tag{2.28}$$

We are going to prove that $\underline{u} \le \tilde{u} \le \overline{u}$, $\underline{v} \le \tilde{v} \le \overline{v}$. In fact, suppose that there exists $(x_0,t_0) \in \Omega\times\mathbb{R}$ such that $\tilde{u}(x_0,t_0) > \overline{u}(x_0,t_0)$. Let $D^* = \{(x,t) \in \Omega\times\mathbb{R} : \tilde{u}(x,t) > \overline{u}(x,t)\}$ and let $\hat{D}$ be the connected component in D^* such that $(x_0,t_0) \in \hat{D}$; then $\hat{D}$ is an open subset of $\Omega\times\mathbb{R}$ and $(\tilde{u} - \overline{u}) = 0$, for all $(x,t) \in \partial\hat{D}$.

If $(x,t) \in \hat{D}$, then

$$\begin{aligned} L[\overline{u} - \tilde{u}](x,t) &+ 2K[\overline{u} - \tilde{u}](x,t) \\ &\ge F_1(x,t,\overline{u}(x,t),\overline{v}(x,t)) \\ &\quad + G_1(x,t,\overline{u}(x,t),\underline{v}(x,t)) + 2K\overline{u}(x,t) \\ &\quad - \big(F_1(x,t,\sigma_1\tilde{u}(x,t),\sigma_2\tilde{v}(x,t)) \\ &\quad + G_1(x,t,\sigma_1\tilde{u}(x,t),\sigma_2\tilde{v}(x,t)) + 2K\sigma_1\tilde{u}(x,t)\big) \end{aligned}$$

$$
\begin{aligned}
&= \big(F_1(x,t,\bar{u}(x,t),\bar{v}(x,t)) - F_1(x,t,\sigma_1\tilde{u}(x,t),\sigma_2\tilde{v}(x,t)) \\
&\qquad\qquad\qquad + K(\bar{u} - \sigma_1\tilde{u})(x,t)\big) \\
&\quad + \big(G_1(x,t,\bar{u}(x,t),\underline{v}(x,t)) - G_1(x,t,\sigma_1\tilde{u}(x,t),\sigma_2\tilde{v}(x,t)) \\
&\qquad\qquad\qquad + K(\bar{u} - \sigma_1\tilde{u})(x,t)\big) \\
&= \big(F_1(x,t,\bar{u}(x,t),\bar{v}(x,t)) - F_1(x,t,\sigma_1\tilde{u}(x,t),\bar{v}(x,t)) \\
&\qquad\qquad\qquad + K(\bar{u} - \sigma_1\tilde{u})(x,t)\big) \\
&\quad + \big(F_1(x,t,\sigma_1\tilde{u}(x,t),\bar{v}(x,t)) - F_1(x,t,\sigma_1\tilde{u}(x,t),\sigma_2\tilde{v}(x,t)\big) \\
&\quad + \big(G_1(x,t,\bar{u}(x,t),\underline{v}(x,t)) - G_1(x,t,\sigma_1\tilde{u}(x,t),\underline{v}(x,t)) \\
&\qquad\qquad\qquad + K(\bar{u} - \sigma_1\tilde{u})(x,t)\big) \\
&\quad + G_1(x,t,\sigma_1\tilde{u}(x,t),\underline{v}(x,t)) - G_1(x,t,\sigma_1\tilde{u}(x,t),\sigma_2\tilde{v}(x,t)\big) \\
&\geq \left(\frac{\partial F_1}{\partial s_1}(x,t,\theta_1(x,t),\bar{v}(x,t)) + K\right)(\bar{u} - \sigma_1\tilde{u})(x,t) \\
&\quad + \left(\frac{\partial G_1}{\partial s_1}(x,t,\theta_2(x,t),\underline{v}(x,t)) + K\right)(\bar{u} - \sigma_1\tilde{u})(x,t) \geq 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \Omega \times \mathbb{R},
\end{aligned}
$$

where $\theta_i(x,t)$, $i = 1,2$ is a point between $\underline{u}(x,t)$ and $\sigma_1\tilde{u}(x,t)$. Since $(\tilde{u} - \bar{u})(x,t) = 0$ for all $(x,t) \in \partial\hat{D}$, the last inequality by the Maximum Principle contradicts the assumption. This proves that $(\tilde{u} - \bar{u})(x,t) \leq 0$, for all $(x,t) \in \Omega \times \mathbb{R}$.

Using the same arguments as above, one can prove that $(\tilde{u} - \underline{u})(x,t) \geq 0$, for all $(x,t) \in \Omega \times \mathbb{R}$ and $\underline{v} \leq \tilde{v} \leq \bar{v}$ on $\Omega \times \mathbb{R}$. This completes the proof of the theorem. ∎

3. APPLICATION TO AUTOCATALYSIS

In this section we apply Theorem 1.2 to the following T-periodic system:

$$
\begin{aligned}
&L[u](x,t) = av(x,t) - (v(x,t))^2 u(x,t) \qquad \text{on } \Omega \times \mathbb{R}, \\
&L[v](x,t) = v(x,t)^2 u(x,t) + f(x,t) \\
&\qquad\qquad - (a+1)v(x,t) \qquad \text{on } \Omega \times \mathbb{R}, \qquad\qquad \text{(VI)} \\
&Bu(x,t) = \psi_1(x,t), \quad Bv(x,t) = \psi_2(x,t) \qquad \text{in } \partial\Omega \times \mathbb{R}, \\
&u(x,t) = u(x,t+T), \quad v(x,t+T) = v(x,t) \qquad \text{in } \overline{\Omega} \times \mathbb{R},
\end{aligned}
$$

where a is a positive constant, $f \in \hat{A}$ and $\psi_i \in \hat{E}$, $\psi_i \geq 0$, $f > 0$, $\psi_i \not\equiv 0$, $i = 1, 2$, $c(x, t) < 0$ in $\overline{\Omega} \times \mathbb{R}$, and B as in last section. It is related to autocatalytic chemical reactions. It is also known as the Brusselator problem, a model for the chemical morphogenetic process due to Turing [12]. This system was studied for the case of initial boundary value problem in [5] and [8].

Let us denote by $F_1(u, v) = av$, $G_1(u, v) = -v^2 u$, $F_2(u, v) = v^2 u$, and $G_2(x, t, u, v) = -(a + 1)v + f(x, t)$. The functions F_i, G_i, $i = 1, 2$ satisfy [H1] and [H3], for $(x, t, u) \in \Omega \times \mathbb{R}^2$, $v \geq 0$.

To state the main result of this section, we denote by $z \in \hat{A}$ the unique solution of the T-periodic linear boundary value problem:

$$\begin{aligned} L[u](x,t) &= f(x,t) \qquad \text{on } \Omega \times \mathbb{R}, \\ Bu(x,t) &= \psi_1(x,t) + \psi_2(x,t) \qquad \text{on } \partial\Omega \times \mathbb{R}, \\ u(x, T+t) &= u(x,t), \qquad \text{on } \overline{\Omega} \times \mathbb{R}. \end{aligned} \tag{3.1}$$

Using a large positive constant and the zero function for comparison, we can apply Theorem 1.1 to show that $z \geq 0$ in $\overline{\Omega} \times \mathbb{R}$.

We will need the following additional assumption:

[H4] There exists a real number $k > 1$ such that

$$\left(\|z\|_\infty^{\Omega \times [0,T]}\right)^2 \leq k^{-1}\left((1 - k^{-1})a + 1\right). \tag{3.2}$$

Note that (3.2) is always satisfied if we let $k = 2a/(a + 1)$ and a is large enough. Thus [H4] is satisfied if $a > 0$ is large enough.

We consider the T-periodic nonlinear boundary value problem:

$$\begin{aligned} L[v] &= -(a+1)v + v^2 k(z - v) + f \qquad \text{on } \Omega \times \mathbb{R}, \\ Bv &= \psi_2 \qquad \text{on } |\Omega \times \mathbb{R}, \\ v(x,t) &= v(x, t+T). \end{aligned} \tag{3.3}$$

LEMMA 3.1. *The T-periodic problem* (3.3) *has a solution $v \in \hat{E}$, $v > 0$ on* $\Omega \times \mathbb{R}$.

Proof. Let $v_1 = z$, where z is the solution of (3.1). Then

$$\begin{aligned} L[v_1] = f &\geq -(a+1)v_1 + v_1^2 k(z - v_1) + f \qquad \text{on } \Omega \times \mathbb{R}, \\ Bv_1 &= \psi_1 + \psi_2 \geq \psi_2 \qquad \text{on } \partial\Omega \times \mathbb{R}. \end{aligned} \tag{3.4}$$

The inequality (3.4) means that v_1 is a supersolution of (3.3).

Let $\delta > 0$ be small enough such that $\lambda_1 \phi\delta + (a+1)\delta\phi \le f$, $z \ge \delta\phi$ on $\Omega \times \mathbb{R}$, where $\lambda_1 > 0$ is the first eigenvalue with eigenfunction $\phi > 0$, $\|\phi\|_\infty = 1$ of the T-periodic problem:

$$\begin{aligned} L[u] &= \lambda u \qquad \text{on } \partial\Omega \times \mathbb{R}, \\ Bu &= 0 \qquad \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,t) &= u(x,t+T) \qquad \text{on } \overline{\Omega} \times \mathbb{R}. \end{aligned}$$

We set $v_2 = \delta\phi$. Then

$$L[v_2] = \lambda_1 \delta\phi \le -(a+1)v_2 + f + v_2^2 k(z - v_2) \qquad \text{on } \Omega \times \mathbb{R}. \tag{3.5}$$

The inequality (3.5) means that v_2 is a subsolution of (3.3), and by (3.4) and (3.5) there exists a solution $\bar{v} \in \hat{E}$ of (3.3) such that $v_2 \le \bar{v} \le v_1$ on $\overline{\Omega} \times \mathbb{R}$.

Notice that $(z - \bar{v}) \ge 0$ on $\Omega \times \mathbb{R}$. ∎

THEOREM 3.1. *If* [H4] *holds, then there exists a solution* (u, v), $u \in \hat{E}$, $v \in \hat{E}$ *of* (VI), *with* $u(x,t) \ge 0$, $v(x,t) > 0$ *on* $\Omega \times \mathbb{R}$.

Proof. We set $\bar{u} = k(z - \bar{v})$, where $\bar{v}$ is defined above. Since by Lemma 3.1 and [H4], $(a+1) \ge k\|z\|_\infty^2 + a/k \ge (\bar{v})k(z - \bar{v}) + a/k$, we have

$$\begin{aligned} L[\bar{u}] &= k(L[z] - L[\bar{v}]) \\ &= k\big((a+1)\bar{v} - (\bar{v})^2 k(z - \bar{v})\big) \\ &\ge a\bar{v} \qquad \text{on } \Omega \times \mathbb{R}. \end{aligned} \tag{3.6}$$

Notice that $\bar{u} \ge 0$ on $\Omega \times \mathbb{R}$, and $B\bar{u} \ge \psi_1$ on $\partial\Omega \times \mathbb{R}$.
Since $\bar{v}$ satisfies Eq. (3.3); it follows that

$$\begin{aligned} L[\bar{v}] &= -(a+1)\bar{v} + (\bar{v})^2 \bar{u} + f \qquad \text{on } \Omega \times \mathbb{R}, \\ B\bar{v} &= \psi_2 \qquad \text{on } \partial\Omega \times \mathbb{R}. \end{aligned} \tag{3.7}$$

Let $\delta_0 > 0$ be small enough such that

$$\lambda_1 \delta_0 \phi + (a+1)\delta_0 \phi \le f \quad \text{and} \quad \delta_0 \phi < \bar{v} \qquad \text{on } \Omega \times \mathbb{R}, \tag{3.8}$$

and set $\underline{v} = \delta_0 \phi$. From (3.8) we obtain

$$\begin{aligned} L[\underline{v}] &\le -(a+1)\underline{v} + f \qquad \text{on } \Omega \times \mathbb{R}, \\ B\underline{v} &= 0 \le \psi_2 \qquad \text{on } \partial\Omega \times \mathbb{R}. \end{aligned} \tag{3.9}$$

Let $\underline{u}$ be the unique solution of the T-periodic linear equation

$$\begin{aligned} L[u] + (\bar{v})^2 u &= a\underline{v} \qquad \text{on } \Omega \times \mathbb{R}. \\ Bu &= \psi_1 \qquad \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,t) &= u(x,t+T) \qquad \text{on } \overline{\Omega} \times \mathbb{R}. \end{aligned} \tag{3.10}$$

By (3.6), (3.10), and comparison, we obtain $0 \le \underline{u} \le \bar{u}$ on $\Omega \times \mathbb{R}$.

It follows from (3.6), (3.7), (3.9), and (3.10) that $(\bar{u}, \bar{v})$, $(\underline{u}, \underline{v})$ are, respectively, the supersolution and subsolution of (VI). The assertion follows from Theorem 1.2. ∎

4. MONOTONE SCHEME FOR APPROXIMATING THE SOLUTIONS

In this section we construct a scheme for approximating the periodic solution for problem (I). We will construct a monotonic sequence of periodic functions converging to an upper bound for the periodic solution from above and another monotonic sequence converging to a lower bound from below. If the limits of the two sequences are the same, then there is a unique periodic solution to the problem (I) between the upper and lower solutions. As in the elliptic case, this method can be used to find conditions for uniqueness in some cases (see, e.g., Section 5.3 in [7]). Since every smooth function can be written as the sum of a nondecreasing function and a nonincreasing function, the scheme is useful for analyzing a more extensive class of problems than it seems in the right side of (I). The scheme can also be readily used for the elliptic case as well.

Assume hypotheses [H1] to [H3]; we now proceed to construct monotonic decreasing sequences $\bar{u}_i, \bar{v}_i$ and monotonic increasing sequences $\underline{u}_i, \underline{v}_i$, $i = 1, 2, \ldots$ such that

$$\begin{aligned} &\underline{u}_i(x,t) \le u(x,t) \le \bar{u}_i(x,t), \\ &\underline{v}_i(x,t) \le v(x,t) \le \bar{v}_i(x,t) \qquad \text{for all } (x,t) \in \Omega \times \mathbb{R}, \end{aligned} \tag{4.1}$$

and each i, where (u, v) is the solution of (I) described in Theorem 1.2.

First, we define $(\bar{u}_0, \underline{u}_0, \bar{v}_0, \underline{v}_0) = (\bar{u}, \underline{u}, \bar{v}, \underline{v})$, whose existence is given by hypothesis [H2]. Let K_1 and K_2 be positive constants so that for fixed q in the interval $[\min_\Omega \underline{v}, \max_\Omega \bar{v}]$, $(x,t) \in \Omega \times \mathbb{R}$, the functions $F_1(x,t,p,q) + K_1 p$ and $G_1(x,t,p,q) + K_2 p$ are nondecreasing in p in the interval $[\min_\Omega \underline{u}, \max_\Omega \bar{u}]$; while for fixed p in the same interval, $(x,t) \in \Omega \times \mathbb{R}$, the functions $F_2(x,t,p,q) + K_1 q$ and $G_2(x,t,p,q) + K_2 q$ are nondecreasing in q in the interval $[\min_\Omega \underline{v}, \max_\Omega \bar{v}]$. Set $K = K_1 + K_2$; we inductively define $(\bar{u}_i, \underline{u}_i, \bar{v}_i, \underline{v}_i) \in \hat{E} \times \hat{E} \times \hat{E} \times \hat{E}$ to be the unique T-periodic solution of the following linear problems:

$$\begin{aligned} L[\bar{u}_i] + K\bar{u}_i &= F_1(x,t,\bar{u}_{i-1},\bar{v}_{i-1}) + G_1(x,t,\bar{u}_{i-1},\underline{v}_{i-1}) \\ &\quad + K\bar{u}_{i-1} \qquad \text{on } \Omega \times \mathbb{R}, \\ L[\underline{u}_i] + K\underline{u}_i &= F_1(x,t,\underline{u}_{i-1},\underline{v}_{i-1}) + G_2(x,t,\underline{u}_{i-1},\bar{v}_{i-1}) \\ &\quad + K\underline{u}_{i-1} \qquad \text{on } \Omega \times \mathbb{R}, \\ B\bar{u}_i(x,t) &= B\underline{u}_i(x,t) = \psi_1(x,t) \qquad \text{in } \partial\Omega \times \mathbb{R}. \end{aligned} \tag{4.2}$$

$$\begin{aligned} L[\bar{v}_i] + K\bar{v}_i &= F_2(x,t,\bar{u}_i,\bar{v}_{i-1}) + G_2(x,t,\underline{u}_i,\bar{v}_{i-1}) \\ &\quad + K\bar{v}_{i-1} \qquad \text{on } \Omega \times \mathbb{R}, \\ L[\underline{v}_i] + K\underline{v}_i &= F_2(x,t,\underline{u}_i,\underline{v}_{i-1}) + G_2(x,t,\bar{u}_i,\underline{v}_{i-1}) \\ &\quad + K\underline{v}_{i-1} \qquad \text{on } \Omega \times \mathbb{R}, \\ B\bar{v}_i(x,t) &= B\underline{v}_i(x,t) = \psi_2(x,t) \qquad \text{in } \partial\Omega \times \mathbb{R}. \end{aligned} \tag{4.3}$$

It is clear from Theorem 1.1 that $(\bar{u}_i, \underline{u}_i \bar{v}_i, \underline{v}_i)$, $i = 1, 2, \dots$ are uniquely defined T-periodic functions in $\hat{E}$ for each component. We next show that these sequences are monotonic.

THEOREM 4.1. *Assume hypotheses* [H1] *to* [H3]. *The sequences defined in* (4.2) *and* (4.3) *satisfy*

$$\begin{aligned} &\underline{u}_0(x,t) \le \underline{u}_1(x,t) \le \underline{u}_2(x,t) \le \cdots \le \bar{u}_2(x,t) \le \bar{u}_1(x,t) \\ &\qquad \le \bar{u}_0(x,t), \\ &\underline{v}_0(x,t) \le \underline{v}_1(x,t) \le \underline{v}_2(x,t) \le \cdots \le \bar{v}_2(x,t) \le \bar{v}_1(x,t) \\ &\qquad \le \bar{v}_0(x,t), \qquad \textit{for } (x,t) \in \bar{\Omega} \times \mathbb{R}. \end{aligned} \tag{4.4}$$

Proof. By hypothesis [H2], we have $\underline{u}_0 \le \bar{u}_0$ and $\underline{v}_0 \le \bar{v}_0$ in $\bar{\Omega} \times \mathbb{R}$. From [H2] and (4.2) we verify that

$$\begin{aligned} L[\bar{u}_0 - \bar{u}_1] + K(\bar{u}_0 - \bar{u}_1) \ge 0 \quad \text{and} \quad & L[\underline{u}_0 - \underline{u}_1] + K(\underline{u}_0 - \underline{u}_1) \\ &\le 0 \qquad \text{in } \bar{\Omega} \times \mathbb{R}. \end{aligned}$$

Thus the maximum principle for the periodic problem (see, e.g., [3]) yields $\bar{u}_0 \ge \bar{u}_1$ and $\underline{u}_0 \le \underline{u}_1$. Furthermore, from (4.2) we obtain

$$\begin{aligned} L[\bar{u}_1 - \underline{u}_1] &+ K(\bar{u}_1 - \underline{u}_1) \\ &= F_1(x,t,\bar{u}_0,\bar{v}_0) - F_1(x,t,\underline{u}_0,\bar{v}_0) + K_1(\bar{u}_0 - \underline{u}_0) \\ &\quad + F_1(x,t,\underline{u}_0,\bar{v}_0) - F_1(x,t,\underline{u}_0,\underline{v}_0) \\ &\quad + G_1(x,t,\bar{u}_0,\underline{v}_0) - G_1(x,t,\underline{u}_0,\underline{v}_0) \\ &\quad + K_2(\bar{u} - \underline{u}_0) + G_1(x,t,\underline{u}_0,\underline{v}_0) \\ &\quad - G_1(x,t,\underline{u}_0,\bar{v}_0) \ge 0 \qquad \text{in } \Omega \times \mathbb{R}, \end{aligned}$$

by the choice of K_1, K_2 and hypothesis [H3]. Hence we have $\bar{u}_1 \ge \underline{u}_1$ by the maximum principle. We have obtained

$$\underline{u}_0 \le \underline{u}_1 \le \bar{u}_1 \le \bar{u}_0 \qquad \text{in } \overline{\Omega} \times \mathbb{R}.$$

By means of the same procedures, we can obtain from hypotheses [H2], [H3], and Eq. (4.3) that

$$\underline{v}_0 \le \underline{v}_1 \le \bar{v}_1 \le \bar{v}_0 \qquad \text{in } \overline{\Omega} \times \mathbb{R}.$$

Suppose that we have shown

$$\underline{u}_0 \le \underline{u}_1 \le \cdots \le \underline{u}_i \le \bar{u}_i \le \cdots \le \bar{u}_1 \le \bar{u}_0 \tag{4.5}$$

$$\underline{v}_0 \le \underline{v}_1 \le \cdots \le \underline{v}_i \le \bar{v}_i \le \cdots \le \bar{v}_1 \le \bar{v}_0 \qquad \text{in } \overline{\Omega} \times \mathbb{R}. \tag{4.6}$$

From Eq. (4.2) we have

$$\begin{aligned} L[\bar{u}_{i+1} - \bar{u}_i] &+ K(\bar{u}_{i+1} - \bar{u}_i) \\ &= F_1(x,t,\bar{u}_i,\bar{v}_i) - F_1(x,t,\bar{u}_{i-1},\bar{v}_i) + K_1(\bar{u}_i - \bar{u}_{i-1}) \\ &\quad + F_1(x,t,\bar{u}_{i-1},\bar{v}_i) - F_1(x,t,\bar{u}_{i-1},\bar{v}_{i-1}) \\ &\quad + G_1(x,t,\bar{u}_i,\underline{v}_i) - G_1(x,t,\bar{u}_{i-1},\underline{v}_i) \\ &\quad + K_2(\bar{u}_i - \bar{u}_{i-1}) + G_1(x,t,\bar{u}_{i-1},\underline{v}_i) \\ &\quad - G_1(x,t,\bar{u}_{i-1},\underline{v}_{i-1}) \\ &\le 0 \qquad \text{in } \Omega \times \mathbb{R}, \end{aligned}$$

by the choice of K_1, K_2, (4.5), (4.6), and hypothesis [H3]. This implies that $\bar{u}_{i+1} \le \bar{u}_i$ in $\overline{\Omega} \times \mathbb{R}$. Similarly, we can show that $\underline{u}_{i+1} \ge \underline{u}_i$ and $\bar{u}_{i+1} \ge \underline{u}_{i+1}$

in $\overline{\Omega} \times \mathbb{R}$, and thus we have

$$\underline{u}_i \leq \underline{u}_{i+1} \leq \bar{u}_{i+1} \leq \bar{u}_i \qquad \text{in } \overline{\Omega} \times \mathbb{R}. \tag{4.7}$$

We next use Eq. (4.3) to obtain

$$\begin{aligned} L[\bar{v}_{i+1} - \bar{v}_i] &+ K(\bar{v}_{i+1} - \bar{v}_i) \\ &= F_2(x,t,\bar{u}_{i+1},\bar{v}_i) - F_2(x,t,\bar{u}_i,\bar{v}_i) + K_1(\bar{v}_i - \bar{v}_{i-1}) \\ &\quad + F_2(x,t,\bar{u}_i,\bar{v}_i) - F_2(x,t,\bar{u}_i,\bar{v}_{i-1}) \\ &\quad + G_2(x,t,\underline{u}_{i+1},\bar{v}_i) - G_2(x,t,\underline{u}_i,\bar{v}_i) \\ &\quad + K_2(\bar{v}_i - \bar{v}_{i-1}) + G_2(x,t,\underline{u}_i,\bar{v}_i) \\ &\quad - G_2(x,t,\underline{u}_i,\bar{v}_{i-1}) \\ &\leq 0 \qquad \text{in } \Omega \times \mathbb{R}, \end{aligned}$$

by the choice of K_1, K_2, (4.5), (4.7), and hypothesis [H3]. This implies that $\bar{v}_{i+1} \leq \bar{v}_i$ in $\overline{\Omega} \times \mathbb{R}$. Similarly, we can deduce that $\underline{v}_{i+1} \geq \underline{v}_i$ and $\bar{v}_{i+1} \geq \underline{v}_{i+1}$ in $\overline{\Omega} \times \mathbb{R}$. We thus obtain

$$\underline{v}_i \leq \underline{v}_{i+1} \leq \bar{v}_{i+1} \leq \bar{v}_i \qquad \text{in } \overline{\Omega} \times \mathbb{R}. \tag{4.8}$$

From (4.7) and (4.8), we deduce by induction that (4.4) is true. ∎

The four monotonic sequences described above converge, and we denote

$$u_* = \lim_{i\to\infty} \underline{u}_i, \quad u^* = \lim_{i\to\infty} \bar{u}_i, \quad v_* = \lim_{i\to\infty} \underline{v}_i, \quad v^* = \lim_{i\to\infty} \bar{v}_i. \tag{4.9}$$

We can now obtain approximations of the periodic solutions found in the last section.

THEOREM 4.2. *Assume hypotheses* [H1] *to* [H3]. *The solution* (u, v), $u, v \in \hat{E}$ *of* (I) *described in Theorem* 1.2 *satisfies*

$$\underline{u} \leq \underline{u}_i \leq u \leq \bar{u}_i \leq \bar{u} \quad \text{and} \quad \underline{v} \leq \underline{v}_i \leq v \leq \bar{v}_i \leq \bar{v} \qquad \text{in } \overline{\Omega} \times \mathbb{R}, \tag{4.10}$$

for each integer $i = 1, 2, \ldots$

Proof. From (4.2) we have

$$\begin{aligned} L[u - \bar{u}_1] + K(u - \bar{u}_1) &= F_1(x,t,u,v) - F_1(x,t,\bar{u},v) + K_1(u - \bar{u}) \\ &\quad + F_1(x,t,\bar{u},v) - F_1(x,t,\bar{u},\bar{v}) \\ &\quad + G_1(x,t,u,v) - G_1(x,t,\bar{u},v) \\ &\quad + K_2(u - \bar{u}) + G_1(x,t,\bar{u},v) \\ &\quad - G_1(x,t,\bar{u},\underline{v}) \\ &\leq 0 \qquad \text{in } \Omega \times \mathbb{R}, \end{aligned}$$

by the choice of K_1, K_2, and hypothesis [H3]. This implies that $u \leq \bar{u}_1$ in $\overline{\Omega} \times \mathbb{R}$. Similarly, we show that $\underline{u}_1 \leq u$. Next we show that $\underline{v}_1 \leq v \leq \bar{v}_1$. Continuing in this manner, we prove inductively as in Theorem 4.1 that the inequalities (4.10) are valid for each positive integer i. ∎

Remark 4.1. Assume hypotheses [H1] to [H3] concerning the functions F_i, G_i and the existence of coupled upper and lower solutions. Let $\bar{u}_i$, $\underline{u}_i$, $\bar{v}_i$, and $\underline{v}_i$, $i = 1, 2, \dots$ be sequences defined by (4.2) and (4.3); and let u_*, u^*, v_*, and v^* be functions as defined in (4.9). Then, in the case where $u^* \equiv u_*$ and $v^* \equiv v_*$ in $\overline{\Omega} \times \mathbb{R}$, problem (I) has a unique periodic solution. Under additional conditions on the nonlinear terms on the right side of (I), this situation can be shown to be true. (See Section 5.3 in [7] for similar elliptic cases, where uniqueness is obtained with this method for a case analogous to $F_1 \equiv G_2 \equiv 0$ and $|\partial G_1/\partial s_2|, |\partial F_2/\partial s_1|$ are relatively small). Of course, more general results require further detailed treatments, which are too lengthy for the present article.

Remark 4.2. In problem (I), the right-hand side is not quasimonotone. Consequently, the convergent schemes described in Chapter 5 in [7] or in [10] are not really applicable to the present problem. The scheme here is different and is applicable to more situations.

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